Journal of Statistical Physics, Vol. 24, No. 1, 1981

Perturbative Analysis of the Convection Instability

C. P. Enz[1]

Received January 8, 1980

A rotationally symmetric Bénard cell is considered with the aim of applying a perturbation formalism which works far from equilibrium. As a first step the unperturbed Gaussian stationary state is constructed from the linearized equations of motion. Then the stationary and the dynamic vertex structures generated by the nonlinear terms are discussed in view of a possible renormalization-group application.

KEY WORDS: Bénard instability; perturbation theory; far from equilibrium; Fokker–Planck description.

1. INTRODUCTION

With renormalization-group methods the fluctuation-dominated, or "critical," region around an instability which is "far from equilibrium" can in principle be treated in analogy with critical dynamics. The main obstacle so far seems to have been the lack of a perturbation scheme general enough to apply to this regime. This difficulty has, however, been overcome in a recent work by Onuki and Kawasaki,[1] who have studied a typical example of a far-from-equilibrium transition, namely a fluid near its critical point subject to a shear flow.

In this paper we study another example of this class, namely a fluid subject to a vertical heat flow near its convection, or Bénard instability. In a Fokker–Planck description of the fluctuations it turns out that a precise meaning can be given to the notion "far from equilibrium," namely that of a stationary state with a nonvanishing dissipative probability current. The latter turns out to be a key quantity for understanding the difficulties with perturbation theory and also to overcome them.

[1] Départment de Physique Théorique, Université de Genève, Geneva, Switzerland.

The convection instability is a classical problem treated exhaustively by Chandrasekhar.[2] But it was Graham who succeeded in showing its analogy with a second-order equilibrium transition.[3,4] More recently, the importance of rotational symmetry in the horizontal plane for the question of whether the transition is first or second order has been emphasized by Swift and Hohenberg.[5] This symmetry will be preserved in the present formulation. As an introductory application, we will briefly discuss the well-known linear deterministic stability analysis which fixes the bifurcation point. The main subject, however, will be the determination of the unperturbed stationary probability distribution, which far from equilibrium turns out to be nontrivial.[1] The Gaussian distribution describing this state will be the basis of our perturbation formalism, which is a generalization of earlier versions[6,7] and therefore will only be sketched. The application of this formalism to renormalization-group calculations will be given elsewhere.

2. THE THERMOHYDRODYNAMIC EQUATIONS

In Boussinesq approximation[2] the dynamics of the fluid are described by the Navier–Stokes equation

$$\dot{\mathbf{u}} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -(1/\rho)\,\nabla p + \nu_0\nabla^2\mathbf{u} + \boldsymbol{\chi} + \boldsymbol{\xi} \tag{1}$$

and by the heat diffusion equation

$$\dot{T} + (\mathbf{u}\cdot\nabla)T = (\kappa/c_v)\,\nabla^2 T - (1/c_v)\,\nabla\cdot\mathbf{q} \tag{2}$$

where $\mathbf{u}(\mathbf{r},t)$ is the velocity field which satisfies the transversality condition

$$\nabla\cdot\mathbf{u} = 0 \tag{3}$$

and $T(\mathbf{r},t)$ is the local temperature, p is the pressure, ρ the mass density, ν_0 the shear viscosity, κ the heat conductivity, c_v the specific heat, and $\boldsymbol{\chi}$ a destabilizing force per unit mass. The heat flux $\mathbf{q}$ and the force $\boldsymbol{\xi}$ are stochastic quantities to be specified later.

We are interested in the deviations $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$ and $v_4 = T - T_0$ from a given stationary flow pattern $\mathbf{u}_0(\mathbf{r})$and $u_4(\mathbf{r}) = T_0(\mathbf{r})$. In these variables and in Cartesian coordinates $\mathbf{r} = (x_1, x_2, x_3)$ Eqs. (1) and (2) may be written in the combined form (repeated indices are summed from 1 to 4)

$$\dot{v}_i = -v_j\nabla_j v_i + \nu_i\nabla^2 v_i - \frac{1}{\rho}\nabla_i p - S_{ij}v_j + \xi_i; \qquad i = 1,\ldots,4 \tag{4}$$

where $\nabla_4 \equiv 0$, $\nu_i = \nu_0$ $(i = 1,2,3)$, $\nu_4 = \kappa/c_v$, $\xi_4 = -c_v^{-1}\nabla\cdot\mathbf{q}$, and

$$S_{ij} = A_{ij} + B_{ij} \tag{5}$$

Here

$$A_{ij}v_j = u_{0j}\,\nabla_j v_i + (\nabla_j u_{0i})v_j \tag{6}$$

describes a drag and

$$B_{ij}(v)v_j = \chi_i(u) - \chi_i(u_0) \tag{7}$$

with $\chi_4 \equiv 0$ the destabilizing force.

In the case of the convection instability the destabilizing force is due to the acceleration of gravity $\mathbf{g} = (0, 0, -g)$ acting through thermal expansion $\delta\rho/\rho = -\alpha(T - \bar{T})$, where $\bar{T}$ is an average temperature, so that

$$\chi = \left[1 - \alpha(T - \bar{T})\right]\mathbf{g} \tag{8}$$

Below the instability, the flow pattern for a Bénard cell bounded by two horizontal planes is given by

$$\mathbf{u}_0 = 0, \qquad u_{04} = T_0 = \bar{T} + \beta x_3 \tag{9}$$

which is a solution of the deterministic equations (1), (2) with $\xi = 0$, $\xi_4 = 0$, and $p_0 = \bar{p} - \rho g x_3(1 - \frac{1}{2}\alpha\beta x_3)$. Equations (6)–(9) lead to a constant matrix (5),

$$S = \begin{pmatrix} O & \gamma\hat{n} \\ \beta\hat{n} & 0 \end{pmatrix} \tag{10}$$

where $\gamma \equiv -\alpha g$ and $\hat{n} = (0, 0, 1)$.

Eliminating the pressure p with the help of Eq. (3) and going over to Fourier components according to

$$v_{i\mathbf{q}} = V^{-1/2}\int d^3r\, v_i(\mathbf{r})\exp(-i\mathbf{q}\cdot\mathbf{r}) \tag{11}$$

V being the volume, Eq. (4) takes the form[7,8]

$$\dot{v}_{\perp i\mathbf{q}} = f^0_{i\mathbf{q}} + f'_{i\mathbf{q}} + g'_{i\mathbf{q}} + \xi_{i\mathbf{q}}; \qquad i = 1, \ldots, 4 \tag{12}$$

where

$$f^0_{i\mathbf{q}} = -V^{-1/2}\sum_{\mathbf{k}} iq_j P_{il}(\mathbf{q})v_{\perp j\mathbf{k}}v_{\perp l\mathbf{q}-\mathbf{k}} \tag{13}$$

is the mode-coupling force,

$$f'_{i\mathbf{q}} = -\nu_i q^2 v_{\perp i\mathbf{q}} \quad \text{(not summed)} \tag{14}$$

is the viscous force, and

$$g'_{i\mathbf{q}} = -P_{il}(\mathbf{q})S_{lj}v_{\perp j\mathbf{q}} \tag{15}$$

is the drag and destabilizing force (primes designate dissipative quantities). In these formulas

$$P_{ij}(\mathbf{q}) = \delta_{ij} - \hat{q}_i\hat{q}_j \tag{16}$$

with $\hat{q} = \mathbf{q}/q$; $\hat{q}_4 \equiv 0$ is the transverse projection, which has the property $P_{il}P_{lj} = P_{ij}$ and $v_{\perp i\mathbf{q}} = P_{ij}(\mathbf{q})v_{j\mathbf{q}}$.

3. LINEAR STABILITY ANALYSIS

For the determination of the bifurcation point only the linear deterministic part of Eq. (12),

$$f'_\mu + g'_\mu = \Lambda_{\mu\nu} v_\nu \tag{17}$$

is needed. Here $\mu = i, \mathbf{q}$, repeated indices being also summed over $\mathbf{q}$, and

$$\Lambda_{i\mathbf{q},j\mathbf{k}} = P_{il}(\mathbf{q})\lambda_{lj}(\mathbf{q})\delta_{\mathbf{q},\mathbf{k}} \tag{18}$$

with

$$\lambda_{ij}(\mathbf{q}) = \nu_i q^2 \delta_{ij} + S_{il} P_{lj}(\mathbf{q}) \tag{19}$$

Since the instability is of the soft-mode type,[2] we also put $\dot{v}_i = 0$, so that Eq. (12) reduces to $\Lambda_{\mu\nu} v_\nu = 0$, or

$$\nu_0 q^2 \mathbf{v}_{\perp\mathbf{q}} + \gamma \mathbf{p} v_{4\mathbf{q}} = 0, \qquad \beta \mathbf{p} \cdot \mathbf{v}_\mathbf{q} + \nu_4 q^2 v_{4\mathbf{q}} = 0 \tag{20}$$

Here $\mathbf{p}(\mathbf{q}) \equiv (P_{13}(\mathbf{q}), P_{23}(\mathbf{q}), P_{33}(\mathbf{q}))$ has the properties $\mathbf{p}^2 = p_3$ and $P_{ij}p_j = p_i$. Choosing the x_1 axis such that $\mathbf{q} = (q_1, 0, q_3)$, the solution of Eqs. (20) is given by

$$q_1 q_3 v_{1\mathbf{q}} - q_1^2 v_{3\mathbf{q}} - (\nu_4/\beta) q^4 v_{4\mathbf{q}} = 0 \tag{21}$$

$v_{2\mathbf{q}} = 0$, and by the condition

$$q^6/q_1^2 = Q^4 \equiv \beta\gamma/\nu_0\nu_4 \equiv R h^{-4} > 0 \tag{22}$$

which means $\beta < 0$. Here R is the Rayleigh number and h the height of the Bénard cell. Eliminating $v_{1\mathbf{q}}$ with the help of the transversality condition (3), Eq. (21) yields

$$v_{4\mathbf{q}} = -(\nu_0 q^4/\gamma q_1^2) v_{3\mathbf{q}} \tag{23}$$

Condition (22) has the solutions $q = q^{(\lambda)}(Q, q_1), q_3 = q_3^{(\lambda)}(Q, q_1) = [(q^{(\lambda)})^2 - q_1^2]^{1/2}$ ($\lambda = 0, \pm$), where

$$q^{(0)} = (Q^2 q_1)^{1/3}, \qquad q^{(\pm)} = (-1 \pm i\sqrt{3})q^{(0)}/2 \tag{24}$$

In order to illustrate the well-known determination of the bifurcation point Q_c, q_{1c},[2] we choose fixed boundary conditions for simplicity,

$$v_i = 0; \qquad i = 1, \ldots, 4; \qquad x_3 = \pm h/2 \tag{25}$$

Then the above solutions are even or odd functions of x_3; the convection instability is known to be given by the even solutions.[2] Hence, reversing the Fourier transform (11), we have[8]

$$v_{3\mathbf{q}}(x_1, x_3) = \sum_{\lambda=0,\pm} A^{(\lambda)}(x_1)\cos\left(q_3^{(\lambda)}x_3\right) \tag{26}$$

with

$$A^{(\lambda)}(x_1) = V^{-1/2}\sum_{q_1} v_{3,q_1,q_3^{(\lambda)}} e^{iq_1x_1} \tag{27}$$

and the boundary conditions $v_3 = 0$, $\nabla_3 v_3 = 0$, $v_4 = 0$ at $x_3 = \pm h/2$, which follow from (25), (3), and $v_2 = 0$, are, respectively,

$$\begin{aligned} \sum_\lambda A^{(\lambda)}(x_1)\cos\left(q_3^{(\lambda)}h/2\right) &= 0 \\ \sum_\lambda A^{(\lambda)}(x_1)q_3^{(\lambda)}\sin\left(q_3^{(\lambda)}h/2\right) &= 0 \\ \sum_\lambda A^{(\lambda)}(x_1)\left(q^{(\lambda)}\right)^{-2}\cos\left(q_3^{(\lambda)}h/2\right) &= 0 \end{aligned} \tag{28}$$

Note that $v_1 = 0$ at $x_3 = \pm h/2$ is determined by the boundary condition $\nabla_3 v_3 = 0$ through Eq. (3). The condition for a nontrivial solution $A^{(0)}$, $A^{(+)}$, $A^{(-)}$ then determines the bifurcation point $Q = Q_c$, $q_1 = q_{1c}$.[2]

4. THE FOKKER–PLANCK EQUATION

In the next step the stochastic quantities $\xi_{i\mathbf{q}}$ are introduced explicitly. They are determined by the correlations

$$\langle \xi_\mu(t)\xi_\nu(t')\rangle = 2C_{\mu\nu}\,\delta(t-t') \tag{29}$$

where $C_{\mu\nu} = C_{\nu\mu}$ is the diffusion matrix, which we write in the form[7]

$$C_{i\mathbf{q},j\mathbf{k}} = D(q)P_{ij}(\mathbf{q})\,\delta_{\mathbf{q}+\mathbf{k},0} \tag{30}$$

where $D(q)$ is a nonnegative function (more generally, two functions D_0 and D_4 should be introduced).

With (29), Eqs. (12) are generalized Langevin equations, which may equivalently be expressed in terms of the Fokker–Planck equation for the probability distribution $P(v, t)$,[6]

$$\dot{P} = -\partial_\mu J_\mu \tag{31}$$

where $\partial_\mu = \partial/\partial v_\mu$,

$$J_\mu = (f_\mu - \partial_\nu C_{\nu\mu})P \tag{32}$$

is the probability flux, and

$$f_\mu = f_\mu^0 + f_\mu' + g_\mu' \tag{33}$$

Correlation functions between the $v_{i\mathbf{q}}(t)$ are averages with the stationary probability distribution, which may be written in the form

$$P^s(v) = Z^{-1}e^{-F(v)} \tag{34}$$

In terms of the free energy $F(v)$ the stationary version of Eq. (31) reads

$$[\partial_\mu - (\partial_\mu F)](j_\mu + f_\mu^0) = 0 \tag{35}$$

where

$$j_\mu = f'_\mu + g'_\mu + C_{\mu\nu}(\partial_\nu F) - (\partial_\nu C_{\nu\mu}) \tag{36}$$

so that $j_\mu P^s$ is the dissipative part of the stationary probability flux. $j_\mu = 0$ are the potential conditions,[6] which express detailed balance and, for $T = \text{const}$, lead to a Maxwell distribution,

$$F = \frac{1}{2T} \sum_{\mathbf{q}} |\mathbf{v}_{\mathbf{q}}|^2$$

However, with a Maxwell distribution $j_\mu = 0$ implies, besides the Einstein relation $D(q) = T\nu_0 q^2$,[7] that $S = 0$, which means, according to (10) and (22), the absence of a soft mode instability. Thus, $S \neq 0$ implies a stationary state with a nonvanishing dissipative probability current, $j_\mu P^s \neq 0$, which is a genuine far-from-equilibrium situation.

5. GAUSSIAN UNPERTURBED STATIONARY STATE

We again turn to the linear zero-frequency situation, but including the stochastic quantities. In the Fokker–Planck version of the problem this leads to Eq. (35) without the nonlinear term f_μ^0. The aim of course is to construct the basis for a perturbation treatment. This means that the unperturbed stationary state $P_{\text{un}}^s = Z_{\text{un}}^{-1}\exp(-F_{\text{un}})$ must be a Gaussian, so that the unperturbed Green's functions have the factorization property of Wick's theorem. Hence we seek a solution of Eq. (35) with

$$F_{\text{un}} = \tfrac{1}{2} E_{\mu\nu} v_\mu v_\nu \tag{37}$$

Making use of Eq. (17), this leads to the condition

$$\sigma_{\mu\nu} - E_{\mu\lambda}\sigma_{\lambda\nu} v_\mu v_\nu = 0 \tag{38}$$

valid for all v_μ. Here

$$\sigma_{\mu\nu} \equiv C_{\mu\lambda} E_{\lambda\nu} - \Lambda_{\mu\nu} \tag{39}$$

and the unperturbed dissipative probability flux is $j_\mu^{\text{un}} P_{\text{un}}^s = \sigma_{\mu\nu} v_\nu P_{\text{un}}^s$.

According to Eqs. (18) and (30), we may look for a solution of the form $E_{i\mathbf{q},j\mathbf{k}} = E_{ij}(\mathbf{q})\,\delta_{\mathbf{q}+\mathbf{k},0}$ and $\sigma_{i\mathbf{q},j\mathbf{k}} = \sigma_{ij}(\mathbf{q})\,\delta_{\mathbf{q},\mathbf{k}}$. In terms of σ_{ij} the condition

(38) becomes, in matrix notation,

$$\sum_{\mathbf{q}} \operatorname{Tr}\sigma = 0; \qquad E\sigma + (E\sigma)^{\mathrm{T}} = 0 \tag{40}$$

where the index T means the transposed, $E^{\mathrm{T}}(\mathbf{q}) = E(-\mathbf{q})$, and

$$\sigma(\mathbf{q}) = P(\mathbf{q})\{D(q)E(\mathbf{q}) - \lambda(\mathbf{q})\} \tag{41}$$

$P(\mathbf{q})$ is the matrix defined by (16),

$$P(\mathbf{q}) = \begin{pmatrix} \mathsf{P}(\mathbf{q}) & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \tag{42}$$

With Eqs. (10), (19), and (41) one easily checks that $\sigma = 0$ is in general not a solution of (40). Hence the state described by E is far from equilibrium in the sense that $j_\mu^{\mathrm{un}} \neq 0$.

Making the simplifying assumption $E^{\mathrm{T}}(\mathbf{q}) = E(\mathbf{q}) = E(-\mathbf{q})$ and introducing the inverse matrix

$$E^{-1} = X = \begin{pmatrix} \mathsf{X} & \mathbf{x} \\ \mathbf{x} & \xi \end{pmatrix} \tag{43}$$

the second condition (40) becomes

$$2DP = P\lambda X + X\lambda^{\mathrm{T}}P \tag{44}$$

Going through the algebra, one finds

$$\xi = A_4 + 2B_4\eta\mathbf{p}^2, \qquad \mathbf{x} = -\eta\mathbf{p} \tag{45}$$

and

$$\tfrac{1}{2}(\mathsf{PX} + \mathsf{XP}) = A_0\mathsf{P} + 2C_0\eta|\mathbf{p}|\mathsf{Q} \tag{46}$$

where

$$\eta \equiv (B_0 + C_4)A_m/\left[1 - 2(C_0 + C_4)B_m\mathbf{p}^2\right] \tag{47}$$

and

$$\mathsf{Q} = (1/|\mathbf{p}|)\mathbf{p} \otimes \mathbf{p} \tag{48}$$

has the properties $\mathsf{Q}^2 = \mathsf{Q}$ and $\mathsf{PQ} = \mathsf{QP} = \mathsf{Q}$. Here the abbreviations

$$A_i = D(q)/\nu_i q^2; \qquad B_i = \beta/2\nu_i q^2; \qquad C_i = \gamma/2\nu_i q^2 \tag{49}$$

have been introduced for $i = 0, 4, m$, where $\nu_m \equiv \frac{1}{2}(\nu_0 + \nu_4)$.

The solution of Eq. (46) is not unique; written in the form

$$\mathsf{X} = c\mathsf{1} + r\mathsf{P} + s\mathsf{Q} \tag{50}$$

one finds $c + r = A_0$ and $s = 2C_0\eta|\mathbf{p}|$. Here $c \neq 0$ since otherwise $\mathsf{PX} = \mathsf{X}$ and hence $\operatorname{Det}\mathsf{X} = 0$. The inversion of X is now readily done and, with the help of the first condition (40), yields a unique matrix E, which, however, we will not give here, the important point being to have shown the existence of a solution (37).

6. SKETCH OF THE PERTURBATION THEORY AND CONCLUSION

Inclusion of the nonlinear term f_μ^0 in Eq. (35) will lead to a perturbed stationary state (34) which may be written as a power series

$$F = F_{\text{un}} + \sum_{N=1}^{\infty} F_N \tag{51}$$

with

$$F_N = \frac{1}{N} \phi^{(N)}_{\mu_1 \cdots \mu_N} v_{\mu_1} \cdots v_{\mu_N} \tag{52}$$

One finds that all orders $N \geqslant 2$ are generated by an external field $\phi_\mu^{(1)}$, while for $\phi_\mu^{(1)} = 0$ the lowest order term is found to be F_5. With this information one may analyze the convection instability by applying static renormalization-group methods for small $|\mathbf{q} - \mathbf{q}_c|$. It is easily seen from examining diagrams that F_5 generates terms F_1 and F_3 in first order and terms F_4 in second order of perturbation theory. It remains to be seen whether the rotational symmetry around the x_3 axis which is implicit in our formulation will cancel F_1 and F_3, as was the case in the iterative solution of Swift and Hohenberg.[5]

The dynamics near the convective instability may also be analyzed by an appropriate perturbation theory, combined with renormalization-group methods for small $|\mathbf{q} - \mathbf{q}_c|$ and ω. Since the instability is of the soft-mode type, $\omega_c = 0$, it is desirable to have a formalism in which stationarity is the zero-frequency limit of dynamics in the sense that zero-time correlation functions are connected to the corresponding zero-frequency response functions by a fluctuation–dissipation theorem. This is guaranteed if a dynamical matrix $D_{\mu\nu}$ with the properties

$$D_{\mu\nu} = D^0_{\mu\nu} - C_{\mu\nu}; \qquad D^0_{\mu\nu} = -D^0_{\nu\mu} \tag{53}$$

and

$$f_\mu = (\partial_\nu F) D_{\nu\mu} - (\partial_\nu D_{\nu\mu}) \tag{54}$$

exists.[6] Similarly to F, such a matrix may be constructed as a power series

$$D_{\mu\nu} = \sum_{N=0}^{\infty} D_{N\mu\nu} \tag{55}$$

with

$$D_{N\mu\nu} = \Delta^{(N)}_{\mu\nu\lambda_1 \cdots \lambda_N} v_{\lambda_1} \cdots v_{\lambda_N}; \qquad N \geqslant 1$$

$$D_{0\mu\nu} = \Delta^{(0)}_{\mu\nu} - C_{\mu\nu} \tag{56}$$

The lowest order nonvanishing terms are found to be $\Delta^{(0)}$ and $\Delta^{(3)}$. The term $\Delta^{(0)}$ leads to the unperturbed conjugate variables $\hat{v}_{0\mu}$ and $\Delta^{(3)}$, etc., give rise to exceptional dynamical vertices[7] of order five and higher, while f_μ^0 gives rise to a normal dynamical vertex[6] of order three.

Thus it becomes apparent that the Feynman diagram formalism developed earlier[6,7] is well suited, in the generalized version sketched here, to analyze the convection instability. This formalism has no obvious resemblance with the one developed by Onuki and Kawasaki,[1] which lacks the fluctuation–dissipation relation mentioned above.

It would be of interest to know whether the method described here also works in the case of a hard-mode instability (limit circle). Since in this case $\omega_c \neq 0$, one would expect a generalization of the fluctuation–dissipation relation to hold in which the correlation functions with period ω_c are the limit $\omega \to \omega_c$ of the corresponding response functions.

There is not much hope, of course, that this type of perturbation formalism would work in the problem of fully developed turbulence, where a renormalized perturbation theory based on a generating functional seems to be necessary.[9]

REFERENCES

1. A. Onuki and K. Kawasaki, *Ann. Phys. (N.Y.)* **121**:456 (1979).
2. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon, Oxford, 1961).
3. R. Graham, *Phys. Rev. Lett.* **31**:1479 (1973); *Phys. Rev. A* **10**:1762 (1974).
4. R. Graham and H. Pleiner, *Phys. Fluids* **18**:130 (1975).
5. J. Swift and P. C. Hohenberg, *Phys. Rev. A* **15**:319 (1977).
6. C. P. Enz, *Physica* **89A**:1 (1977).
7. C. P. Enz, *Physica* **94A**:20 (1978).
8. M. Droz, C. P. Enz, and A. Malaspinas, in *Stochastic Processes in Nonequilibrium Systems*, L. Garrido, P. Seglar, and P. J. Shepherd, eds. (Lecture Notes in Physics, Vol. 84; Springer, Berlin, 1978), p. 280.
9. H. Horner and R. Lipowsky, *Z. Physik B* **33**:223 (1979).